

Contents

Part A Number Theory

1	Highlights in the History of Number Theory: 1700 BC–2008	3
1.1	Early Roots to Fermat	3
1.2	Fermat	6
1.2.1	Fermat's Little Theorem	7
1.2.2	Sums of Two Squares	7
1.2.3	Fermat's Last Theorem	8
1.2.4	Bachet's Equation	9
1.2.5	Pell's Equation	9
1.2.6	Fermat Numbers.....	9
1.3	Euler.....	10
1.3.1	Analytic Number Theory	11
1.3.2	Diophantine Equations	12
1.3.3	Partitions.....	13
1.3.4	The Quadratic Reciprocity Law	14
1.4	Lagrange.....	15
1.4.1	Pell's Equation	15
1.4.2	Sums of Four Squares	15
1.4.3	Binary Quadratic Forms	16
1.5	Legendre.....	17
1.6	Gauss' <i>Disquisitiones Arithmeticae</i>	18
1.6.1	Introduction.....	18
1.6.2	Quadratic Reciprocity	18
1.6.3	Binary Quadratic Forms	19
1.6.4	Cyclotomy	20
1.7	Algebraic Number Theory.....	20
1.7.1	Reciprocity Laws.....	20
1.7.2	Fermat's Last Theorem	21
1.7.3	Dedekind's Ideals	22
1.7.4	Summary	23

1.8	Analytic Number Theory	23
1.8.1	The Distribution of Primes Among the Integers: Introduction	24
1.8.2	The Prime Number Theorem	24
1.8.3	The Riemann Zeta Function	25
1.8.4	Primes in Arithmetic Progression	25
1.8.5	More on the Distribution of Primes	26
1.9	Fermat's Last Theorem	27
1.9.1	Work Prior to That of Wiles	27
1.9.2	Andrew Wiles	28
	References	29
2	Fermat: The Founder of Modern Number Theory	31
2.1	Introduction	31
2.2	Fermat's Intellectual Debts	32
2.3	Fermat's Little Theorem and Factorization	33
2.3.1	A Look Ahead	35
2.4	Sums of Squares	36
2.4.1	A Look Ahead	37
2.5	Fermat's Last Theorem	38
2.5.1	A Look Ahead	40
2.6	The Bachet and Pell Equations	40
2.6.1	Bachet's Equation	40
2.6.2	A Look Ahead	41
2.6.3	Pell's Equation	42
2.6.4	A Look Ahead	43
2.7	Conclusion	44
	References	44
3	Fermat's Last Theorem: From Fermat to Wiles	47
3.1	Introduction	47
3.2	The First Two Centuries	48
3.3	Sophie Germain	49
3.4	Lamé	50
3.4.1	Pythagorean Triples	50
3.4.2	Lamé's Proof	51
3.5	Kummer	51
3.6	Early Decades of the Twentieth Century	53
3.7	Several Results Related to FLT, 1973–1993	54
3.8	Some Major Ideas Leading to Wiles' Proof of FLT	55
3.8.1	Elliptic Curves	55
3.8.2	Number Theory and Geometry	55
3.8.3	The Shimura-Taniyama Conjecture	56
3.9	Andrew Wiles	58
3.10	Tributes to Wiles	61
3.11	Is There Life After FLT?	62
	References	63

Part B Calculus/Analysis

4	History of the Infinitely Small and the Infinitely Large in Calculus, with Remarks for the Teacher	67
4.1	Introduction	67
4.2	Seventeenth-Century Predecessors of Newton and Leibniz	68
4.2.1	Introduction	68
4.2.2	Cavalieri	69
4.2.3	Fermat	69
4.3	Newton and Leibniz: The Inventors of Calculus	71
4.3.1	Introduction	71
4.3.2	Didactic Observation	72
4.3.3	Newton	72
4.3.4	Leibniz	75
4.3.5	Didactic Observation	77
4.4	The Eighteenth Century: Euler	78
4.4.1	Introduction	78
4.4.2	Didactic Observation	78
4.4.3	The Algebraization of Calculus	79
4.4.4	Didactic Observation: Discovery and Proof	80
4.5	Foundational Issues in the Seventeenth and Eighteenth Centuries	81
4.5.1	Introduction	81
4.5.2	Newton and Leibniz	82
4.5.3	Berkeley and d'Alembert	84
4.5.4	Euler	85
4.5.5	Lagrange	85
4.6	Calculus Becomes Rigorous: Cauchy, Dedekind, and Weierstrass	87
4.6.1	Introduction	87
4.6.2	Cauchy	87
4.6.3	Dedekind and Weierstrass	91
4.6.4	Didactic Observation	93
4.7	The Twentieth Century: The Nonstandard Analysis of Robinson	94
4.7.1	Introduction	94
4.7.2	Hyperreal Numbers	95
4.7.3	Wider Implications	97
4.7.4	Robinson and Leibniz	97
4.7.5	Didactic Observation	98
	References	99
5	A Brief History of the Function Concept	103
5.1	Introduction	103
5.2	Precalculus Developments	104

5.3	Euler's <i>Introductio in Analysin Infinitorum</i>	105
5.4	The Vibrating-String controversy	106
5.5	Fourier Series	110
5.6	Dirichlet's Concept of Function	112
5.7	"Pathological" Functions	115
5.8	Baire and Analytically Representable Functions	117
5.9	Debates About the Nature of Mathematical Objects	119
5.10	Recent Developments	121
	References	123
6	More on the History of Functions, with Remarks on Teaching	125
6.1	Introduction	125
6.2	Anticipations of the Function Concept.....	126
6.2.1	Babylonian Mathematics.....	126
6.2.2	Greek Mathematics.....	126
6.2.3	The Latitude of Forms	127
6.2.4	Precalculus Developments	127
6.2.5	The Calculus of Newton and Leibniz	128
6.2.6	Remark on Teaching	128
6.3	The Emergence and Consolidation of the Function Concept	128
6.3.1	Remarks on Teaching	130
6.4	Functions Represented by Power Series	130
6.4.1	Remarks on Teaching	133
6.5	Functions Defined by Integrals	134
6.5.1	Remarks on Teaching	135
6.6	Functions Defined as Solutions of Differential Equations	135
6.6.1	Remark on Teaching	137
6.7	Partial Differential Equations and the Representation of Functions by Fourier Series	138
6.7.1	Remarks on Teaching	141
6.8	Functions and Continuity	142
6.8.1	Remarks on Teaching	145
6.9	Conceptual Aspects of Functions	145
6.9.1	Remarks on Teaching	147
6.10	Analytically Representable Functions	147
6.10.1	Remark on Teaching	148
6.11	Conclusion	149
	References.....	149
Part C Proof		
7	Highlights in the Practice of Proof: 1600 BC–2009	153
7.1	Introduction	153
7.2	The Babylonians	153
7.3	Greek Axiomatics.....	154

7.4	Symbolic Notation	156
7.4.1	Leibniz	156
7.4.2	Euler	157
7.5	The Calculus of Cauchy	158
7.6	The Calculus of Weierstrass	161
7.7	The Reemergence of the Axiomatic Method	164
7.8	Foundational Issues	167
7.8.1	Introduction	167
7.8.2	Logicism	168
7.8.3	Formalism	169
7.8.4	Intuitionism	170
7.9	The Era of the Computer	172
	References	177
8	Paradoxes: What Are They Good For?	181
8.1	Introduction	181
8.2	Numbers	182
8.2.1	Incommensurables	182
8.2.2	Negative Numbers	183
8.2.3	Complex Numbers	183
8.3	Logarithms	184
8.4	Functions	186
8.4.1	The Eighteenth Century	186
8.4.2	Nineteenth-Century Views	187
8.5	Continuity	188
8.5.1	Euler and Cauchy	188
8.5.2	Continuity and Differentiability	189
8.6	Aspects of Calculus Other than Continuity	190
8.6.1	Tangents	190
8.6.2	Infinite Series	191
8.7	Sets	192
8.8	Curves	193
8.9	Decomposition of Geometric Objects	194
8.9.1	Doubling the Cube	194
8.9.2	Squaring the Circle	194
8.10	Conclusion	194
	References	195
9	Principle of Continuity: Sixteenth–Nineteenth Centuries	197
9.1	Introduction	197
9.2	Analysis	198
9.2.1	Leibniz and Robinson	199
9.2.2	Euler and Cauchy	200
9.3	Algebra	202
9.3.1	British Symbolical Algebra	203
9.3.2	Cubic Equations and Complex Numbers	205

9.4	Geometry	206
	9.4.1 Projective Geometry	206
	9.4.2 What Is Geometry?.....	209
9.5	Number Theory	209
	9.5.1 The Bachet Equation	210
	9.5.2 Fermat's Last Theorem	211
9.6	Conclusion	211
	References.....	213
10	Proof: A Many-Splendored Thing	215
10.1	Introduction	215
10.2	Heuristics vs. Rigor	216
	10.2.1 Ancient Mathematics.....	216
	10.2.2 Calculus.....	216
	10.2.3 Riemann and Weierstrass	218
10.3	Analysis vs. Synthesis	219
	10.3.1 Ancient Greece	219
	10.3.2 Leibniz and Newton	220
	10.3.3 Eighteenth and Nineteenth Centuries	220
10.4	Pure vs. Applied	221
	10.4.1 Introduction.....	221
	10.4.2 The Vibrating-String Problem	222
	10.4.3 The Heat-Conduction Problem	223
10.5	Legitimate vs. Illegitimate.....	224
	10.5.1 The Quaternions.....	224
	10.5.2 Functions	224
	10.5.3 Continuity	225
	10.5.4 Definitions in Mathematics	225
	10.5.5 Abstraction	226
10.6	Idealists vs. Empiricists.....	226
	10.6.1 Ideals.....	226
	10.6.2 "Pathological" Functions	227
	10.6.3 Invariants	227
	10.6.4 Weyl and Von Neumann	228
10.7	Short vs. Long Proofs.....	228
10.8	Humans vs. Machines	229
10.9	Deterministic vs. Probabilistic Proofs	230
10.10	Theorems vs. Proofs	230
10.11	The Recent Debate.....	231
	References.....	234

Part D Courses Inspired by History

11	Numbers as a Source of Mathematical Ideas	239
11.1	Introduction	239

11.2	Beyond the Complex Numbers	240
11.2.1	A Brief History of "Standard" Number Systems	240
11.2.2	The Quaternions	240
11.2.3	Other Hypercomplex Systems	242
11.2.4	What is a Number?	243
11.3	The Algebraic-Transcendental Dichotomy	243
11.3.1	Introduction	243
11.3.2	Algebraic Numbers	244
11.3.3	Transcendental Numbers	244
11.3.4	Algebraic Numbers and Diophantine Equations	245
11.4	Transfinite Numbers	246
11.4.1	Introduction	246
11.4.2	Some Implications of Cantor's Work	247
11.5	The Personality of Numbers	248
11.6	One, Two, Many	250
11.7	Discovery (Invention), Use, Understanding, Justification	250
11.8	Numbers and Geometry	252
11.9	Numbers and Analysis	254
11.9.1	The Arithmetization of Analysis	255
11.9.2	Nonstandard Analysis	255
11.9.3	Number Theory	256
	References	256
12	History of Complex Numbers, with a Moral for Teachers	261
12.1	Introduction	261
12.2	Birth	261
12.3	Growth	263
12.4	Maturity	265
12.5	The Moral	267
12.6	Projects	270
	References	271
13	A History-of-Mathematics Course for Teachers, Based on Great Quotations	273
13.1	Introduction	273
13.2	What Is Mathematics?	274
13.3	Non-Euclidean Geometry	277
13.4	The Infinite	279
13.5	The Twentieth Century: Foundational Issues	280
13.6	Conclusion	282
	References	282
14	Famous Problems in Mathematics	285
14.1	Introduction	285
14.2	The Themes	285
14.2.1	The Origin of Concepts, Results, and Theories	285

14.2.2	The Roles of Intuition vs. Logic	286
14.2.3	Changing Standards of Rigor	286
14.2.4	The Roles of the Individual vs. the Environment	286
14.2.5	Mathematics and the Physical World	287
14.2.6	The Relativity of Mathematics	287
14.2.7	Mathematics: Discovery or Invention?	287
14.3	The Problems	288
14.3.1	Problem 1: Diophantine Equations	288
14.3.2	Problem 2: Distribution of Primes Among the Integers	290
14.3.3	Problem 3: Polynomial Equations	293
14.3.4	Problem 4: Are There Numbers Beyond the Complex Numbers?	293
14.3.5	Problem 5: Why Is $(-1)(-1)=1$?	294
14.3.6	Problem 6: Euclid's Parallel Postulate	296
14.3.7	Problem 7: Uniqueness of Representation of a Function in a Fourier Series	296
14.3.8	Problem 8: Paradoxes in Set Theory	296
14.3.9	Problem 9: Consistency, Completeness, Independence	297
14.4	Other Problems	297
14.5	General Remarks on the Course	298
	References	298

Part E Brief Biographies of Selected Mathematicians

15	The Biographies	305
15.1	Richard Dedekind (1831–1916)	305
15.1.1	Introduction	305
15.1.2	Life	305
15.1.3	Algebraic Numbers	307
15.1.4	Real Numbers	308
15.1.5	Natural Numbers	308
15.1.6	Other Work	309
15.1.7	Conclusion	310
	References	311
15.2	Leonhard Euler (1707–1783)	312
15.2.1	Introduction	312
15.2.2	Life	313
15.2.3	Analysis	314
15.2.4	Number Theory	317
15.2.5	Conclusion	319
	References	319
15.3	Carl Friedrich Gauss (1777–1855)	320
15.3.1	Life	320

15.3.2	<i>Disquisitiones Arithmeticae</i>	321
15.3.3	Biquadratic Reciprocity	322
15.3.4	Differential Geometry	323
15.3.5	Probability and Statistics	324
15.3.6	The Diary	324
15.3.7	Personality	325
15.3.8	Conclusion	325
	References	326
15.4	David Hilbert (1862–1943)	326
15.4.1	Introduction	326
15.4.2	Life	327
15.4.3	Invariants	328
15.4.4	Algebraic Numbers	329
15.4.5	Foundations of Geometry	331
15.4.6	Analysis and Physics	332
15.4.7	Foundations of Mathematics	332
15.4.8	Mathematical Problems	333
15.4.9	Conclusion	334
	References	335
15.5	Karl Weierstrass (1815–1897)	335
15.5.1	Life	335
15.5.2	Foundations of Real Analysis	336
15.5.3	Complex Analysis	338
15.5.4	Other Work	338
15.5.5	Conclusion	340
	References	341
	Index	343