# Studies in Logic

Mathematical Logic and Foundations

Volume 40

# The Lambda Calculus, its Syntax and Semantics

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