

Newton's *Principia* for the Common Reader

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*I rejoice to concur with the common reader; for
by the common sense of readers, uncorrupted by
literary prejudices, after all the refinements of
subtilty and the dogmatism of learning, must be
generally decided all claim to poetical honours.*

Dr Samuel Johnson

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