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– D, Dual D, Eq, DPC –

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– **S4, S5, S4Grz, T, B, K, QT, MSI, ML, G, G*** –

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– L_3 , L_n , L_\aleph , K_3 , G_3 , G_n , G_\aleph , $S5$ –

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