

ENTAILMENT

THE LOGIC OF RELEVANCE AND NECESSITY

by

ALAN ROSS ANDERSON

and

NUEL D. BELNAP, JR.

and

J. MICHAEL DUNN

with contributions by

KIT FINE

ALASDAIR URQUHART

and further contributions by

DANIEL COHEN

STEVE GIAMBRONE

DOROTHY L. GROVER

ANIL GUPTA

GLEN HELMAN

ERROL P. MARTIN

MICHAEL A. McROBBIE

STUART SHAPIRO

and including a Bibliography of Entailment by

ROBERT G. WOLF

VOLUME II

PRINCETON UNIVERSITY PRESS

CONTENTS

VOLUME I

Analytical Table of Contents	ix
Preface	xxi
Acknowledgments	xxix
I. THE PURE CALCULUS OF ENTAILMENT	3
II. ENTAILMENT AND NEGATION	107
III. ENTAILMENT BETWEEN TRUTH FUNCTIONS	150
IV. THE CALCULUS E OF ENTAILMENT	231
V. NEIGHBORS OF E	339
Appendix: Grammatical propaedeutic	473
Bibliography for Volume I	493
Indices to Volume I	517

VOLUME II

Analytical Table of Contents	ix
Preface	xvii
Acknowledgments	xix
Summary Review of Volume I	xxiii
VI. THE THEORY OF ENTAILMENT	3
VII. INDIVIDUAL QUANTIFICATION	70
VIII. ACKERMANN'S <i>streng</i> Implikation	129
IX. SEMANTICS	142
X. PROOF THEORY AND DECIDABILITY	267
XI. FUNCTIONS, ARITHMETIC, AND OTHER SPECIAL TOPICS	392
XII. APPLICATIONS AND DISCUSSION	488
Bibliography of Entailment (by Robert G. Wolf)	565
Indices to Volume II	711

ANALYTICAL TABLE OF CONTENTS

VOLUME II

SUMMARY REVIEW OF VOLUME I

§R1. Grammatical review	xxiii	
§R2. Axiomatic review	xxiv	
§R3. Natural deduction review	xxv	
VI. THE THEORY OF ENTAILMENT	3	
§30. Propositional quantifiers	3	
§30.1. Motivation	3	
§30.2. Notation	7	
§31. Natural deduction: $E^{v\exists p}$	9	
§31.1. Universal quantification	10	
§31.2. Existential quantification	14	
§31.3. Distribution of universality over disjunction		16
§31.4. Necessity	16	
§31.5. $E^{v\exists p}$ and its neighbors: Summary	18	
§32. $E^{v\exists p}$ and its neighbors: Summary and equivalence		19
§33. Truth values	25	
§33.1. TV^{vp}	26	
§33.2. For every individual x , x is president of the United States between 1850 and 1857	28	
§33.3. E_{fde} and truth values	29	
§33.4. Truth-value quantifiers	31	
§33.5. $R^{v\exists p}$ and TV	32	
§34. First degree entailments in $E^{v\exists p}$ (by Dorothy L. Grover)		33
§34.1. The algebra of first degree entailments of $E^{v\exists p}$		33
§34.2. A consistency theorem	36	
§34.3. Provability theorems	37	
§34.4. Completeness and decidability	45	
§35. Enthymemes	45	
§35.1. Intuitionistic enthymemes	46	
§35.2. Strict enthymemes	47	
§35.3. Enthymematic implication in $E^{v\exists p}$	50	
§35.4. Summary	53	
§36. Enthymematic implications: Embedding H and $S4$ in $E^{v\exists p}$		55
§36.1. H in $E^{v\exists p}$	55	
§36.1.1. Under translation, $E^{v\exists p}$ contains at least H		57

	§36.1.2. Under translation, $E_+^{v\exists p}$ contains no more than H	60	
	§36.2. H and $S4_+$ in $E^{v\exists p}$	62	
§37.	Miscellany	64	
	§37.1. Prenex normal forms (in $T^{v\exists p}$)	64	
	§37.2. The weak falsehood of $\forall p\forall q(p \rightarrow q \rightarrow p)$	66	
	§37.3. $R^{v\exists p}$ is not a conservative extension of $R_+^{v\exists p}$	67	
	§37.4. Definitions of connectives in R with propositional quantifiers	68	
VII.	INDIVIDUAL QUANTIFICATION	70	
§38.	$R^{v\exists x}$, $E^{v\exists x}$, and $T^{v\exists x}$	70	
	§38.1. Natural deduction formulations	71	
	§38.2. Axiomatic formulations and equivalence	72	
§39.	Classical results in first-order quantification theory	73	
	§39.1. Gödel completeness theorem	73	
	§39.2. Löwenheim-Skolem theorem	81	
	§39.3. Gentzen's cut elimination theorem	84	
§40.	Algebra and semantics for first degree formulas with quantifiers	87	
	§40.1. Complete intensional lattices	88	
	§40.2. Some special facts about complete intensional lattices	96	
	§40.3. The theory of propositions	99	
	§40.4. Intensional models	103	
	§40.5. Branches and trees	107	
	§40.6. Critical models	111	
	§40.7. Main theorems	114	
	§40.7.1. Quantificational sequences	114	
	§40.7.2. Quantifier-free sequences	115	
§41.	Undecidability of monadic first degree formulas	117	
§42.	Extension of (γ) to $R^{v\exists x}$ et al.	119	
	§42.1. Terminology for logics and theories	120	
	§42.2. The Way Up	123	
	§42.3. The Way Down	126	
	§42.4. Admissibility of (γ) in $R^{v\exists x}$ et al.	127	
§43.	Miscellany	128	
VIII.	ACKERMANN'S <i>strenge Implikation</i>	129	
§44.	Ackermann's Σ -systems	129	
	§44.1. Motivation	129	
	§44.2. ΣE	131	
	§44.3. ΣE contains E	132	
	§44.4. E contains ΣE	134	
§45.	Σ' , Π' , Π'' , and E (historical)	134	
	§45.1. f goes	137	
	§45.2. (δ) goes	138	

§45.3.	(γ) goes	138	
§46.	Miscellany	139	
	§46.1. Ackermann on strict "implication"		139
	§46.2. An interesting matrix	141	
IX.	SEMANTICS	142	
§47.	Semilattice semantics for relevance logics (by Alasdair Urquhart)	142	
	§47.1. Semantics for R_{\perp}	142	
	§47.2. Semantics for E_{\perp}	146	
	§47.3. Semantics for T_{\perp}	147	
	§47.4. Variations on a theme	149	
§48.	Relational semantics for relevance logics		155
	§48.1. Algebraic vs. set-theoretical semantics		155
	§48.2. Set-theoretical semantics for first degree relevant implications	158	
	§48.3. Three-termed relational (Routley-Meyer) semantics for R_{+}	161	
	§48.4. Strong completeness for R_{+}	169	
	§48.5. Relational semantics for all of R		170
	§48.6. Relational semantics for E	171	
	§48.7. Relational semantics for T, RM , etc.		172
	§48.8. Spinoffs from relational semantics		173
	§48.9. Relational semantics for quantifiers		175
§49.	Binary relational semantics for the mingle systems RM and RM^{v3x}		176
	§49.1. Binary relational semantics for RM		176
	§49.1.1. The binary semantics	177	
	§49.1.2. Informal interpretation	178	
	§49.1.3. Semantical soundness	179	
	§49.1.4. Semantical completeness	181	
	§49.1.5. Decidability by filtration	184	
	§49.1.6. RM models and Sugihara matrices		184
	§49.1.7. The binary semantics with "star operation"		185
	§49.1.8. Limitations of the binary semantics		187
	§49.2. Quantification and RM	188	
	§49.2.1. Grammar and proof theory of RM^{vx}		188
	§49.2.2. Semantics	189	
	§49.2.3. Soundness	190	
	§49.2.4. Completeness of RM^{vx}		190
§50.	Intuitive semantics for first degree entailments and "coupled trees"	193	
	§50.1. Introduction	194	
	§50.2. Relevantly coupled trees		195
	§50.3. Intuitive semantics	197	

§50.4.	Coupled trees and the semantics	203
§50.5.	Tautological entailments and the semantics	203
§50.6.	An earlier semantical gloss of essentially the same mathematics	205
§50.7.	Ruminations	208
§51.	Models for entailment: Relational-operational semantics for relevance logics (by Kit Fine)	208
§51.1.	Models	209
§51.2.	Logics	212
§51.3.	The minimal logic	213
§51.4.	The systems E and R	217
§51.5.	Alternative models	222
§51.6.	Finite models	226
§51.7.	Admissibility of (γ)	229
§52.	No fit between constant-domain semantics and $\mathbf{R}^{\forall\exists x}$	231
§53.	Semantics for quantified relevance logic (by Kit Fine)	235
§53.1.	Models	239
§53.2.	Truth	245
§53.3.	The logics	253
§53.4.	Soundness	254
§53.5.	Completeness	255
§54.	$\mathbf{KR}_{\rightarrow, \&}$: A conjunction-arrow fragment corrupted by Boolean structure	262
§54.1.	Axioms for $\mathbf{KR}_{\rightarrow, \&}$ and their consistency	263
§54.2.	Completeness	264
X.	PROOF THEORY AND DECIDABILITY	267
§60.	Relevant analytic tableaux (with Michael A. McRobbie)	267
§60.1.	The tableau systems	267
§60.2.	Equivalence via left-handed consecution calculuses	274
§60.3.	Problems	278
§61.	A consecution calculus for positive relevant implication with necessity (with Anil Gupta)	279
§61.1.	History	279
§61.2.	Postulates for \mathbf{L} (= $\mathbf{LR}_{\rightarrow}^{\square, \circ}$)	281
§61.3.	Translation and equivalence	282
§61.4.	Some definitions and the normality property	284
§61.5.	Elimination theorem: Outline of proof	287
§61.6.	Closure under substitution and case 1.2	288
§62.	Display logic	294
§62.1.	Introduction	294
§62.2.	Grammar	296
§62.2.1.	Indices and families	296
§62.2.2.	Formula-connectives and structure-connectives	297

§62.2.3.	Formulas, structures, and consecutions	299
§62.2.4.	Interpretation	299
§62.3.	Postulates for DL	300
§62.3.1.	Identity axioms	300
§62.3.2.	Display-equivalence	300
§62.3.3.	Connective postulates	302
§62.3.4.	Reduction	303
§62.4.	Subformula and elimination theorems	305
§62.4.1.	Analysis, parameter, congruence	306
§62.4.2.	Conditions on an analysis	307
§62.4.3.	Proofs of subformula and elimination theorems	310
§62.5.	Some families and logics	313
§62.5.1.	Boolean family and two-valued logic	314
§62.5.2.	Relevant implication	314
§62.5.3.	Entailment	316
§62.5.4.	Ticket entailment	319
§62.5.5.	Semantics of relevance logics	320
§62.5.6.	Modal logics	320
§62.5.7.	Intuitionist logic	324
§62.5.8.	Interfamilial relations	326
§62.6.	Further developments	327
§62.6.1.	Demarcation	327
§62.6.2.	Quantifiers	328
§62.6.3.	Interpolation	328
§62.6.4.	Algebra	328
§62.6.5.	Other connectives	328
§62.6.6.	Restricted rules	329
§62.6.7.	Incompatibility	330
§62.6.8.	Binary structuring and infinite premiss sets	331
§62.6.9.	Priority of the right?	332
§63.	Decidability: Survey	332
§63.1.	Decidability of fragments limited by degrees	333
§63.2.	Decidability of fragments limited by connectives	334
§63.3.	Decidability of neighbors	335
§64.	Which entailments entail which entailments?	336
§64.1.	Reducibility of the decision question to the second degree	337
§64.2.	The positive case	337
§64.3.	The case with negation	344
§65.	The undecidability of all principal relevance logics (by Alasdair Urquhart)	348
§65.1.	Relevant implication and projective geometry	348
§65.1.1.	Models for relevance logics	349

§65.1.2. The logic KR	349	
§65.1.3. Projective spaces	350	
§65.1.4. Model structures constructed from projective spaces	353	
§65.1.5. Undecidability	354	
§65.1.6. More geometrical ruminations	357	
§65.2. The undecidability of entailment and relevant implication	358	
§65.2.1. Introduction	358	
§65.2.2. Coordinate frames in ordered monoids		359
§65.2.3. The algebra of relevance logics	364	
§65.2.4. De Morgan monoids and vector spaces		369
§65.2.5. Undecidability	371	
§65.2.6. Further undecidability results	374	
§66. Minimal logic again (by Errol P. Martin)	375	
§66.1. Three-valued metalogic	376	
§66.2. S-models	377	
§66.3. Reduced valuations	379	
§66.4. The guarded merge theorem	380	
§66.5. Powers's conjecture	382	
§66.6. Significance of all this	384	
§67. Decision procedures for contractionless relevance logics (by Steve Giambrone)	384	
§67.1. Introduction	385	
§67.2. LTW^*_+ and LRW^*_+	385	
§67.3. Vanishing t	386	
§67.4. Denesting	388	
§67.5. Reduction	388	
§67.6. Degree and decidability	389	
§67.7. EW^*_+	391	
XI. FUNCTIONS, ARITHMETIC, AND OTHER SPECIAL TOPICS		392
§70. Functions that really depend on their arguments		392
§70.1. Mathematical concept of dependence	393	
§70.2. Semantic and syntactic concepts of dependence		397
§70.3. Church's λ -I-calculus and Scott's strictness		399
§71. Relevant implication and relevant functions (by Glen Helman)	402	
§71.1. Terms and proofs	403	
§71.2. Relevant abstraction and monadic relevant functions		410
§71.3. Pairing and conjunction	414	
§71.4. Polyadic relevant functions	420	
§72. Relevant Peano arithmetic	423	
§72.1. Postulates for relevant Peano arithmetic	424	
§72.2. Strength and weakness of the extensional fragment		426

§72.3.	Relevant implications or material "implications"?	429
§72.4.	Oddments	433
§73.	Relevant Robinson arithmetic	434
§73.1.	Robinson's axioms	435
§73.2.	$Q_R = Q$	435
§73.3.	$Q_R(1) \neq Q(1)$	440
§73.4.	The relations among R^* , $Q_R(0)$, and $Q_R(1)$	442
§73.5.	Remarks and speculations	443
§74.	Relevant predication: The formal theory	445
§74.1.	Introduction	445
§74.2.	Properties (monadic)	447
§74.3.	Lambda conversion	448
§74.4.	Factor	449
§74.5.	Indiscernibility of identicals	450
§74.6.	Relevant predication	453
§74.7.	Relations (polyadic)	454
§74.8.	Formal consequences of the definitions	456
§74.9.	Background	464
§74.10.	Philosophical applications	468
§74.11.	Technical appendix	469
§75.	Relevant implication and conditional assertion (by Daniel Cohen)	472
§75.1.	Assertivity functions	473
§75.2.	Axiomatization	474
§75.3.	Semantics	476
§75.4.	Soundness	477
§75.5.	Completeness	478
§75.6.	Quantification	486
XII.	APPLICATIONS AND DISCUSSION	488
§80.	Entailment and the disjunctive syllogism	488
§80.1.	Tautological entailment	488
§80.1.1.	Review	488
§80.1.2.	The disjunctive syllogism	488
§80.1.3.	Relevance logic and relevantism	489
§80.1.4.	Our plan	490
§80.2.	Boolean negation	490
§80.2.1.	Background	490
§80.2.2.	A dilemma	492
§80.2.3.	Horn 1	494
§80.2.4.	Horn 2	495
§80.2.5.	A puzzle	497
§80.3.	Relevant arguments for the admissibility of the disjunctive syllogism	498
§80.3.1.	Readings	498

§80.3.2.	“Equivalent” forms	501	
§80.3.3.	Extensional admissibility is useless for a relevantist	502	
§80.4.	The phenomenology of relevantism	502	
§80.4.1.	I’m all right, Jack	503	
§80.4.2.	The relevantist/deductivist parallel	503	
§80.4.3.	The leap of faith	504	
§80.4.4.	The toe in the water	504	
§80.4.5.	The true relevantist	505	
§81.	A useful four-valued logic: How a computer should think		506
§81.1.	The computer	506	
§81.2.	Part 1. Atomic inputs	510	
§81.2.1.	Atomic sentences and the approximation lattice A_4	510	
§81.2.2.	Compound sentences and the logical lattice L_4	513	
§81.2.3.	Entailment and inference: The four-valued logic	518	
§81.2.4.	Observations	520	
§81.3.	Part 2. Compound truth-functional inputs	524	
§81.3.1.	Epistemic states	524	
§81.3.2.	More approximation lattices	527	
§81.3.3.	Formulas as mappings: A new kind of meaning	529	
§81.3.4.	More observations	531	
§81.3.5.	Quantifiers again	532	
§81.4.	Part 3. Implicational inputs and rules	533	
§81.4.1.	Implicational inputs	534	
§81.4.2.	Rules and information states	539	
§81.4.3.	Closure	541	
§82.	Rescher’s hypothetical reasoning: An amended amendment		541
§82.1.	HR-consequence	542	
§82.2.	Objections	544	
§82.3.	Candidate amendments	546	
§82.4.	Conjunctive containment	550	
§83.	Relevance logic in computer science (by Stuart C. Shapiro)		553
§83.1.	Use of the proof theory	554	
§83.1.1.	SWM	555	
§83.1.1.1.	Rules of inference of SWM	556	
§83.1.1.2.	Example	559	
§83.1.2.	Implementations	560	
§83.2.	Use of the four-valued semantics of R	561	