

TABLE OF CONTENTS

PART I

- 1-5. Following a rule (Cf. *Philosophical Investigations* §§ 189-90 ff.).—Steps determined by a formula (1-2). Continuation of a series (3). Inexorability of mathematics; mathematics and truth (4-5). Remark on measuring (5).
- 6-23. Logical inference.—The word "all"; inference from ' $(x).fx$ ' to ' fa ' (10-16). Inference and truth (17-23).
- 24-74. Proof.—Proof as pattern or model (paradigm). Example: hand and pentacle (25 ff.). Proof as picture of an experiment (36). Example: 100 marbles (36 ff.). Construction of figures out of their parts (42-72). Mathematical surprise. Proof and conviction. Mathematics and essence (32, 73, 74). The depth of the essence: the deep need for the convention (74).
- 75-105. Calculation and experiment.—The 'unfolding' of mathematical properties. Example of 100 marbles (75, 86, 88). Unfolding the properties of a polygon (76), and of a chain (79, 80, 91, 92). Measuring (93, 94). Geometrical examples (96-98). Internal properties and relations (102-105); examples from the logic of colour.
- 106-112. Mathematical belief.
- 113-141. Logical compulsion.—In what sense does logical argument compel? (113-117). The inexorability of logic compared with that of the law (118). The 'logical machine' and the kinematics of rigid bodies' (119-125). 'The hardness of the logical *must*' (121). The machine as symbolizing its way of working (122). The employment of a word grasped in a flash (123-130). Possibility as a shadow of reality (125). The employment of a word misunderstood and interpreted as a queer process (127). (For 122-130 see also *Philosophical Investigations* §§ 191-197.) The laws of logic as 'laws of thought' (131-133). Going wrong in a calculation (134-136). Remark on measuring (139). Logical impossibility (140). "What we are supplying is really remarks on the natural history of man" (141).

- 142-155. Foundation of a calculating procedure and of a logical inference.—Calculating without propositions (142-144). Example: sale of timber (142-151). “Are our laws of inference eternal and unalterable”? (154). Logic precedes truth (155).
- 156-169. Mathematics, logic, and experience.—Proof and experiment (156-161). What in mathematics is logic: it moves in the rules of our language (164). The mathematician is an inventor, not a discoverer (167).

Appendix I

- 1-4. Kinds of proposition.—Arithmetic done without propositions (4).
- 5-7. Truth and provability in the system of *Principia Mathematica*.
- 8-19. Discussion of a proposition ‘*P*’ which asserts its own unprovability in the system of *Principia Mathematica*.—The role of contradiction in the language-game (11-14, 17).
20. The propositions of logic. ‘Proposition’ and ‘propositional formation’.

Appendix II

- 1-3. The diagonal procedure.—The concept ‘non-denumerable’ (2). Comparison of the concepts of real number and cardinal number (3).
4. The sickness of a time.
5. Discussion of the proposition “There is no greatest cardinal number”.
- 6-7. Irrational numbers.
- 8-9. \aleph_0 .
- 10-13. Discussion of the proposition “Fractions cannot be arranged in order of magnitude”.
14. Comparison of different games.
- 15-16. Discussion of the proposition that the fractions (number-pairs) can be arranged in an infinite series.
17. The word “infinite”.
18. Finitism, Behaviourism. General remarks.

PART II

- 1-2. Proof.—Mathematical proof must be perspicuous. Role of definitions (2).
- 3-8. Russell's logic and the idea of the reduction of arithmetic to symbolic logic.—The application of the calculation must take care of itself (4). Proof in the Russellian calculus, in the decimal calculus, and in the stroke calculus.
- 9-11. Proof.—Proof as a memorable picture (9). The reproduction of the pattern of a proof (10-11).
- 12-20. Russell's logic and the problem of the mutual relation of different calculating techniques.—What is the invention of the decimal system? (12). Proof in the Russellian calculus and in the decimal system (13). Signs for numbers that can, and that cannot, be taken in (16). Relation of abbreviated and unabbreviated calculating techniques to one another (17-20).
- 21-44. Proof.—Identity and reproducibility of a proof (21). The proof as model. Proof and experiment (22-24). Proof and mathematical conviction (25-26). In the proof we have won through to a decision (27). The proved proposition as a rule. It serves to shew us what it makes *sense* to say (28). The propositions of mathematics as 'instruments of language' (29). The proof introduces a new concept (31). What concept does ' $p \supset p$ ' produce? (32). ' $p \supset p$ ' as pivot of the linguistic method of representation (33). The proof as part of an institution (36). Importance of the distinction between determining and using a sense (37). Acceptance of a proof; the 'geometrical' conception of proof (38-40). The proof as adoption of a particular employment of signs (41). "The proof must be a procedure plain to view" (42). "Logic as foundation of mathematics does not work if only because the cogency of the logical proof stands and falls with its geometrical cogency" (43). In mathematics we can get away from logical proofs (44).
- 45-64. Russell's logic.—Relation between the ordinary and the Russellian technique of proof (45). Criticism of the conception of logic as the 'foundation' of mathematics. Mathematics is a *motley* of calculating techniques. The abbreviated technique as a new aspect of the unabbreviated (46-48). Remark on trigonometry (50). The decimal notation is independent of calculation

with unit strokes (51). Why Russell's logic does not teach us to *divide* (52). Why mathematics is not logic (53). Recursive proof (54). Proof and experiment (55). The correspondence of different calculi; stroke notation and decimal notation (56-57). Several proofs of one and the same proposition; proof and the sense of a mathematical proposition (58-62). The *exact* correspondence of a convincing transition in music and in mathematics (63).

65-76. Calculation and experiment.—Are the propositions of mathematics anthropological propositions? (65). Mathematical propositions conceived as prophecies of concordant results of calculating (66). Agreement is part of the phenomenon of calculating (67). If a calculation is an experiment, what in that case is a mistake in calculation? (68). The calculation as an experiment and as a *path* (69). A proof subserves mutual understanding. An experiment presupposes it (71). Mathematics and the science of conditional calculating reflexes (72). The concept of calculation excludes confusion (75-76).

77-90. Contradiction.—A game in which the one who moves first must always win (77). Calculating with $(a - a)$. The chasms in a calculus are not there if I do not see them (78). Discussion of the heterological paradox (79). Contradiction regarded from the point of view of the language-game. Contradiction as a 'hidden sickness' of the calculus (80). Contradiction and the usability of a calculus (81). The consistency proof and the misuse of the idea of *mechanical* insurance against contradiction (82-89). "My aim is to change the *attitude* towards contradiction and the consistency proof." (82) The role of the proposition: "I must have miscalculated"—the key to understanding the 'foundations' of mathematics (90).

PART III

1-7. On axioms.—The self-evidence of axioms (1-3). Self-evidence and use (2-3). Axiom and empirical proposition (4-5). The negation of an axiom (5). The mathematical proposition stands on four legs, not on three (7).

8-9. Following a rule.—Description by means of a rule (8).

10. The arithmetical assumption is not tied to experience.

- 11-13. The conception of arithmetic as the natural history of numbers.
—Judging experience by means of the picture (12).
14. External relation of the logical (mathematical) proposition.
- 15-19. The possibility of doing applied mathematics without pure mathematics.—Mathematics need not be done in *propositions*; the centre of gravity may lie in *action* (15). The commutative law as an example (16-17).
20. Calculation as a mechanical activity.
21. The picture as a proof.
- 22, 27. Intuition.
26. What is the difference between *not* calculating and calculating *wrong*?
- 29-33. Proof and mathematical concept formation.—Proof alters concept formation. Concept formation as the limit of the empirical (29). Proof does not *compel*, but *guides* (30). Proof conducts our experience into definite channels (31, 33). Proof and prediction (33).
34. The philosophical problem is: how can we tell the truth and at the same time *pacify* these strong prejudices?
- 35-36. The mathematical proposition.—We acknowledge it *by* turning our back on it (35). The effect of the proof: one plunges into the new rule (36).
- 39, 42. Synthetic character of mathematical propositions.—The distribution of primes as an example (42).
40. The result set up as equivalent to the operation.
41. That proof must be perspicuous means that causality plays no part in proof.
- 43-44. Intuition in mathematics.
47. The mathematical proposition as determination of a concept, following upon the discovery of a new form.
48. The working of the mathematical machine is only the *picture* of the working of a machine.
49. The picture as a proof.
- 50-51. Reversal of a word.

- 52-53. Mathematical and empirical propositions.—The assumption of a mathematical concept expresses the confident expectation of certain experiences; but the establishment of this measure is not equivalent to the expression of the expectations (53).
- 55-60. Contradiction.—The liar (58). Contradiction conceived as something supra-propositional, as a monument with a Janus head enthroned above the propositions of logic (59).

PART IV

- 1-4. Mathematics as a game and as a machine-like activity.—Does the calculating machine calculate? (2). How far is it necessary to have a concept of 'proposition' in order to understand Russell's mathematical logic? (4).
- 5-8. Is a misunderstanding about the possible application of mathematics any objection to a calculation as part of mathematics?—Set theory (7).
- 9-13. The law of excluded middle in mathematics.—Where there is nothing to base a decision on, we must invent something in order to give the application of the law of excluded middle a sense.
- 14-16 and 21-23. 'Alchemy' of the concept of infinity and of other mathematical concepts whose application is not understood.—Infinite predictions (23).
- 17-20. The law of excluded middle. The mathematical proposition as a commandment. Mathematical existence.
- 24-27. Existence proofs in mathematics.—"The harmful invasion of mathematics by logic" (24; see also 46 and 48). The mathematically general does not stand to the mathematically particular in the same relation as does the general to the particular elsewhere (25). Existence proofs which do not admit of the construction of what exists (26-27).
28. Proof by *reductio ad absurdum*.
- 29-40. On the extensional and intensional in mathematics; Dedekind's theorem without irrational numbers (30). How does this theorem come by its deep content? (31). The picture of the number line (32, 37). Discussion of the concept of a cut (33-34). Generality in the realm of functions is an unordered generality (38). Dis-

cussion of the mathematical concept of a function; extension and intension in analysis (39-40).

41. Concepts occurring in 'necessary' propositions must also have a meaning in non-necessary ones.
- 42-46. On proof and understanding of a mathematical proposition.—The proof conceived as a *movement* from one concept to another (42). Understanding a mathematical proposition (45-46). The proof introduces a new concept. The proof serves to convince one of something (45). Existence proof and construction (46).
47. A concept is not essentially a predicate.
48. 'Mathematical logic' has completely blinded the thinking of mathematicians and philosophers.
49. The numerical sign goes along with the sign for a concept and only together with this is it a measure.
50. On the concept of generality.
51. The proof shews *how* the result is yielded.
- 52-53. General remarks.—The philosopher is the man who has to cure himself of many sicknesses of the understanding before he can reach the notions of the healthy human understanding.

PART V

1. The role of propositions that treat of measures and are not empirical propositions. Such a proposition (e.g. 12 inches = 1 foot) is embedded in a technique, and so in the conditions of this technique; but it is not a statement of those conditions.
2. The role of a rule. It can also be used to make predictions. This depends on properties of the measuring rods and of the people who use them.
3. A mathematical proposition—a transformation of the expression. The rule considered from the point of view of usefulness and from that of dignity. How are two arithmetical expressions supposed to say the same thing? They are made equivalent in arithmetic.
4. Someone who learns arithmetic simply by following my examples. If I say, "If you do with these numbers what I did for you with the others, you will get such-and-such a result"—this seems to be both a prediction and a mathematical proposition.

5. Does not the contrast between rules of description and descriptive propositions shade off on every side?
6. What is common to a mathematical proposition and a mathematical proof, that they should both be called "mathematical"?
The proof as a picture. It is not assent alone which makes this picture into a calculation, but the consensus of assents.
7. Does the sense of the proposition change when a proof has been found? The new proof gives the proposition a place in a new system.
8. Russell's ' $\sim f(f)$ '.
Let us say we have got some of our results because of a hidden contradiction. Does that make them illegitimate?
Might we not let a contradiction stand?
9. "A method for avoiding a contradiction mechanically." It is not bad mathematics that is amended here, but a new bit of mathematics is invented.
10. Must logical axioms always be convincing?
11. The people who sometimes reduce by expressions of value 0.
12. If the calculation has lost its point for me, as soon as I know that I can get any arbitrary result from it—did it have no point as long as I did *not* know this?
One thinks that contradiction *has* to be senseless.
13. What does mathematics need a foundation for?
A good angel will always be necessary.
14. The practical value of calculating. Calculation and experiment.
A calculation as part of the technique of an experiment.
The activity of calculating can also be an experiment.
15. Is mathematics supposed to bring facts to light? Does it not take mathematics to determine the character of what we call a 'fact'? Does it not teach us to ask about certain facts?
In mathematics there are no causal connexions, only the connexions of the pattern.
16. Remarks.
17. The network of joins in a wall. Why do we call this a mathematical problem?
Does mathematics make experiments with *units*?

18. "The proposition that says of itself that it is unprovable"—how is this to be understood?
19. The construction of a propositional sign from axioms according to rules; it appears that we have demonstrated the actual sense of the proposition to be false, and at the same time proved it.
20. Calculation and experience.
21. Does the "heterological" contradiction shew a logical character of this concept?
22. A game. And after a certain move any attempt to go on playing proves to be against the rules.
23. Logical inference is part of a language game.
 Logical inference and non-logical inference.
 The rules of logical inference can be neither wrong nor right. They determine the meaning of the signs.
24. A reasonable procedure with numerals need not be what we call "calculating".
25. Is not a mathematics with an application that is sheer fantasy, still mathematics?
26. The formation of concepts may be essential to a great part of mathematics; and have no role in other parts.
27. A people who do not notice a contradiction, and draw conclusions from it.
 Can it be a mathematical task to make mathematics into mathematics?
28. If a contradiction were actually found in arithmetic, this would show that an arithmetic with such a contradiction can serve us very well.
29. "The class of lions is not a lion, but the class of classes is a class."
30. "I always lie." What part might this sentence play in human life?
31. Logical inference. Is not a rule something arbitrary?
 "It is impossible for human beings to recognize an object as different from itself."

32. "Correct—i.e. it conforms to the rule."
33. "Bringing the *same*"—how can I explain this to someone?
34. When should we speak of a proof of the existence of '777' in an expansion?
35. "Concept formation" may mean various things.
The concept of a rule for forming an infinite decimal.
-

36. Is it essential to the concept of calculating, that people generally reach this result?
37. If I ask, e.g., whether a certain body moves according to the equation of a parabola—what does mathematics do in this case?
38. Questions about the way in which mathematics forms concepts.
-

39. Can one not make mathematical experiments after all?
40. Adding shapes. Possibilities in folding a piece of paper. Suppose we did not separate geometrical and physical possibility?
Might not people in certain circumstances calculate with figures, without a particular result's *having* to come out?
If the calculation shews you a causal connexion, you are not calculating.
Mathematics is normative.
41. The introduction of a new rule of inference as a transition to a new language game.
42. Observation that a surface is red and blue, but not that it is red.
Inferences from this.
Can logic tell us what we must observe?
43. A surface with stripes of changing colours.
Could implications be observed?
44. Someone says he sees a red and yellow star, but not anything yellow.
45. "I hold to a rule."

46. The mathematical *must*—the expression of an attitude towards the technique of calculating.
The expression of the fact that mathematics forms concepts.
47. The case of seeing the complex formed from A and B, but seeing neither A nor B.
Can I see A and B, but only observe $A \vee B$?
And *vice versa*.
48. Experiences and timeless propositions.
49. In what sense can a proposition of arithmetic be said to give us a concept?
50. Not every language-game contains something that we want to call a "concept".
51. Proof and picture.