

Table of Contents

<i>Preface</i>	xvii
Mathematical Background	xix
Selecting Among the Chapters	xx
<i>Acknowledgments</i>	xxi
<i>Notational Conventions</i>	xxiii
1. Introduction	
1.1 THREE BASIC PROCEDURES OF FUNDAMENTAL MEASUREMENT	1
1.1.1 Ordinal Measurement	2
1.1.2 Counting of Units	3
1.1.3 Solving Inequalities	5
1.2 THE PROBLEM OF FOUNDATIONS	6
1.2.1 Qualitative Assumptions: Axioms	6
1.2.2 Homomorphisms of Relational Structures: Representation Theorems	8
1.2.3 Uniqueness Theorems	9
1.2.4 Measurement Axioms as Empirical Laws	13
1.2.5 Other Aspects of the Problem of Foundations	13
1.3 ILLUSTRATIONS OF MEASUREMENT STRUCTURES	13
1.3.1 Finite Weak Orders	14
1.3.2 Finite, Equally Spaced, Additive Conjoint Structures	17

1.4	CHOOSING AN AXIOM SYSTEM	21
1.4.1	Necessary Axioms	21
1.4.2	Nonnecessary Axioms	23
1.4.3	Necessary and Sufficient Axiom Systems	24
1.4.4	Archimedean Axioms	25
1.4.5	Consistency, Completeness, and Independence	26
1.5	EMPIRICAL TESTING OF A THEORY OF MEASUREMENT	26
1.5.1	Error of Measurement	27
1.5.2	Selection of Objects in Tests of Axioms	28
1.6	ROLES OF THEORIES OF MEASUREMENT IN THE SCIENCES	31
1.7	PLAN OF THE BOOK	33
	EXERCISES	35
2.	Construction of Numerical Functions	
2.1	REAL-VALUED FUNCTIONS ON SIMPLY ORDERED SETS	38
2.2	ADDITIVE FUNCTIONS ON ORDERED ALGEBRAIC STRUCTURES	43
2.2.1	Archimedean Ordered Semigroups	44
2.2.2	Proof of Theorem 4 (Outline)	46
2.2.3	Preliminary Lemmas	47
2.2.4	Proof of Theorems 4 and 4' (Details)	48
2.2.5	Archimedean Ordered Groups	53
2.2.6	Note on Hölder's Theorem	53
2.2.7	Archimedean Ordered Semirings	54
2.3	FINITE SETS OF HOMOGENEOUS LINEAR INEQUALITIES	59
2.3.1	Intuitive Explanation of the Solution Criterion	59
2.3.2	Vector Formulation and Preliminary Lemmas	61
2.3.3	Proof of Theorem 7	66
2.3.4	Topological Proof of Theorem 7	67
	EXERCISES	69
3.	Extensive Measurement	
□3.1	INTRODUCTION	71
3.2	NECESSARY AND SUFFICIENT CONDITIONS	72
3.2.1	Closed Extensive Structures	72
3.2.2	The Periodic Case	75
3.3	PROOFS	77
3.3.1	Consistency and Independence of the Axioms of Definition 1	77
3.3.2	Preliminary Lemmas	77
3.3.3	Theorem 1	80

□ 3.4	SUFFICIENT CONDITIONS WHEN THE CONCATENATION OPERATION IS NOT CLOSED	81
3.4.1	Formulation of the Non-Archimedean Axioms	82
3.4.2	Formulation of the Archimedean Axiom	83
3.4.3	The Axiom System and Representation Theorem	84
3.5	PROOFS	85
3.5.1	Consistency and Independence of the Axioms of Definition 3	85
3.5.2	Preliminary Lemmas	86
3.5.3	Theorem 3	87
□ 3.6	EMPIRICAL INTERPRETATIONS IN PHYSICS	87
3.6.1	Length	87
3.6.2	Mass	89
3.6.3	Time Duration	89
3.6.4	Resistance	90
3.6.5	Velocity	91
3.7	ESSENTIAL MAXIMA IN EXTENSIVE STRUCTURES	92
3.7.1	Nonadditive Representations	92
3.7.2	Simultaneous Axiomatization of Length and Velocity	94
3.8	PROOFS	96
3.8.1	Consistency and Independence of the Axioms of Definition 5	96
3.8.2	Theorem 6	96
3.8.3	Theorem 7	98
□ 3.9	ALTERNATIVE NUMERICAL REPRESENTATIONS	99
3.10	CONSTRUCTIVE METHODS	102
3.10.1	Extensive Multiples	103
3.10.2	Standard Sequences	105
3.11	PROOFS	106
3.11.1	Theorem 8	106
3.11.2	Preliminary Lemmas	107
3.11.3	Theorem 9	109
3.12	CONDITIONALLY CONNECTED EXTENSIVE STRUCTURES	111
3.12.1	Thermodynamic Motivation	111
3.12.2	Formulation of the Axioms	113
3.12.3	The Axiom System and Representation Theorem	114
3.12.4	Statistical Entropy	116
3.13	PROOFS	117
3.13.1	Preliminary Lemmas	117
3.13.2	A Group-Theoretic Result	119
3.13.3	Theorem 10	120
3.13.4	Theorem 11	121

3.14 EXTENSIVE MEASUREMENT IN THE SOCIAL SCIENCES	123
3.14.1 The Measurement of Risk	124
3.14.2 Proof of Theorem 13	128
□3.15 LIMITATIONS OF EXTENSIVE MEASUREMENT	130
EXERCISES	132
4. Difference Measurement	
□4.1 INTRODUCTION	136
4.1.1 Direct Comparison of Intervals	137
4.1.2 Indirect Comparison of Intervals	141
4.1.3 Axiomatization of Difference Measurement	143
□4.2 POSITIVE-DIFFERENCE STRUCTURES	145
4.3 PROOF OF THEOREM 1	148
□4.4 ALGEBRAIC-DIFFERENCE STRUCTURES	150
4.4.1 Axiom System and Representation Theorem	151
4.4.2 Alternative Numerical Representations	152
4.4.3 Difference-and-Ratio Structures	152
4.4.4 Strict Inequalities and Approximate Standard Sequences	155
4.5 PROOFS	157
4.5.1 Preliminary Lemmas	157
4.5.2 Theorem 2	158
4.5.3 Theorem 3	158
4.6 CROSS-MODALITY ORDERING	164
4.7 PROOF OF THEOREM 4	166
4.8 FINITE, EQUALLY SPACED DIFFERENCE STRUCTURES	167
4.9 PROOFS	168
4.9.1 Preliminary Lemma	168
4.9.2 Theorem 5	169
4.10 ABSOLUTE-DIFFERENCE STRUCTURES	170
4.11 PROOFS	174
4.11.1 Preliminary Lemmas	174
4.11.2 Theorem 6	176
4.12 STRONGLY CONDITIONAL DIFFERENCE STRUCTURES	177
4.13 PROOFS	184
4.13.1 Preliminary Lemmas	184
4.13.2 Theorem 7	188
EXERCISES	195

5. Probability Representations

□5.1	INTRODUCTION	199
□5.2	A REPRESENTATION BY UNCONDITIONAL PROBABILITY	202
5.2.1	Necessary Conditions: Qualitative Probability	202
5.2.2	The Nonsufficiency of Qualitative Probability	205
5.2.3	Sufficient Conditions	206
5.2.4	Preference Axioms for Qualitative Probability	208
5.3	PROOFS	211
5.3.1	Preliminary Lemmas	211
5.3.2	Theorem 2	212
5.4	MODIFICATIONS OF THE AXIOM SYSTEM	214
5.4.1	QM-Algebra of Sets	214
5.4.2	Countable Additivity	215
5.4.3	Finite Probability Structures with Equivalent Atoms	216
5.5	PROOFS	217
5.5.1	Structure of QM-Algebras of Sets	217
5.5.2	Theorem 4	218
5.5.3	Theorem 6	220
5.6	A REPRESENTATION BY CONDITIONAL PROBABILITY	220
5.6.1	Necessary Conditions: Qualitative Conditional Probability	222
5.6.2	Sufficient Conditions	224
5.6.3	Further Discussion of Definition 8 and Theorem 7	225
5.6.4	A Nonadditive Conditional Representation	227
5.7	PROOFS	228
5.7.1	Preliminary Lemmas	228
5.7.2	An Additive Unconditional Representation	232
5.7.3	Theorem 7	233
5.7.4	Theorem 8	236
5.8	INDEPENDENT EVENTS	238
5.9	PROOF OF THEOREM 10	241
	EXERCISES	243

6. Additive Conjoint Measurement

□6.1	SEVERAL NOTIONS OF INDEPENDENCE	245
6.1.1	Independent Realization of the Components	246
6.1.2	Decomposable Structures	247
6.1.3	Additive Independence	247
6.1.4	Independent Relations	248
□6.2	ADDITIVE REPRESENTATION OF TWO COMPONENTS	250
6.2.1	Cancellation Axioms	250
6.2.2	Archimedean Axiom	253

6.2.3	Sufficient Conditions	254
6.2.4	Representation Theorem and Method of Proof	257
6.2.5	Historical Note	259
6.3	PROOFS	261
6.3.1	Independence of the Axioms of Definition 7	261
6.3.2	Theorem 1	262
6.3.3	Preliminary Lemmas for Bounded Symmetric Structures	262
6.3.4	Theorem 2	264
□6.4	EMPIRICAL EXAMPLES	267
6.4.1	Examples from Physics	267
6.4.2	Examples from the Behavioral Sciences	268
6.5	MODIFICATIONS OF THE THEORY	271
6.5.1	Omission of the Archimedean Property	271
□6.5.2	Alternative Numerical Representations	273
6.5.3	Transforming a Nonadditive Representation into an Additive One	273
6.5.4	Subtractive Structures	274
6.5.5	Need for Conjoint Measurement on $B \subset A_1 \times A_2$	275
6.5.6	Symmetries of Independent and Dependent Variables	276
6.5.7	Alternative Factorial Decompositions	278
6.6	PROOFS	279
6.6.1	Preliminary Lemmas	279
6.6.2	Theorem 3	280
6.6.3	Theorem 4	281
6.6.4	Theorem 6	282
6.7	INDIFFERENCE CURVES AND UNIFORM FAMILIES OF FUNCTIONS	283
6.7.1	A Curve Through Every Point	285
6.7.2	A Finite Number of Curves	286
6.8	PROOFS	288
6.8.1	Theorem 7	288
6.8.2	Theorem 8	289
6.8.3	Preliminary Lemmas About Uniform Families	290
6.8.4	Theorem 9	292
6.9	BISYMMETRIC STRUCTURES	293
6.9.1	Sufficient Conditions	293
6.9.2	A Finite, Equally Spaced Case	297
6.10	PROOFS	298
6.10.1	Theorem 10	298
6.10.2	Theorem 11	300
□6.11	ADDITIVE REPRESENTATION OF n COMPONENTS	301
6.11.1	The General Case	301
6.11.2	The Case of Identical Components	303

6.12 PROOFS	306
6.12.1 Preliminary Lemma	306
6.12.2 Theorem 13	307
6.12.3 Theorem 14	309
6.12.4 Theorem 15	310
6.13 CONCLUDING REMARKS	311
EXERCISES	312
7. Polynomial Conjoint Measurement	
□7.1 INTRODUCTION	316
□7.2 DECOMPOSABLE STRUCTURES	317
7.2.1 Necessary and Sufficient Conditions	318
7.2.2 Proof of Theorem 1	320
□7.3 POLYNOMIAL MODELS	321
7.3.1 Examples	321
7.3.2 Decomposability and Equivalence of Polynomial Models	325
7.3.3 Simple Polynomials	327
7.4 DIAGNOSTIC ORDINAL PROPERTIES	329
□7.4.1 Sign Dependence	329
7.4.2 Proofs of Theorems 2 and 3	335
□7.4.3 Joint-Independence Conditions	339
□7.4.4 Cancellation Conditions	340
□7.4.5 Diagnosis for Simple Polynomials in Three Variables	345
7.5 SUFFICIENT CONDITIONS FOR THREE-VARIABLE SIMPLE POLYNOMIALS	347
7.5.1 Representation and Uniqueness Theorems	348
7.5.2 Heuristic Proofs	350
7.5.3 Generalizations to Four or More Variables	353
7.6 PROOFS	356
7.6.1 A Preliminary Result	356
7.6.2 Theorem 4	357
7.6.3 Theorem 5	361
7.6.4 Theorem 6	364
EXERCISES	366
8. Conditional Expected Utility	
□8.1 INTRODUCTION	369
□8.2 A FORMULATION OF THE PROBLEM	372
8.2.1 The Primitive Notions	372
8.2.2 A Restriction on \mathcal{D}	375
8.2.3 The Desired Representation Theorem	376

8.2.4	Necessary Conditions	376
8.2.5	Nonnecessary Conditions	379
8.2.6	The Axiom System and Representation Theorem	380
8.3	PROOFS	382
8.3.1	Preliminary Lemmas	382
8.3.2	Theorem 1	385
8.4	TOPICS IN UTILITY AND SUBJECTIVE PROBABILITY	391
8.4.1	Utility of Consequences	391
8.4.2	Relations Between Additive and Expected Utility	393
8.4.3	The Consistency Principle for the Utility of Money	395
8.4.4	Expected Utility and Risk	398
8.4.5	Relations Between Subjective and Objective Probability	400
8.4.6	A Method for Estimating Subjective Probabilities	400
8.5	PROOFS	401
8.5.1	Theorem 3	401
8.5.2	Theorem 4	403
8.5.3	Theorem 5	404
8.5.4	Theorem 6	405
8.5.5	Theorem 7	406
□8.6	OTHER FORMULATIONS OF RISKY AND UNCERTAIN DECISIONS	406
8.6.1	Mixture Sets and Gambles	407
8.6.2	Propositions as Primitives	411
8.6.3	Statistical Decision Theory	412
8.6.4	Comparison of Statistical and Conditional Decision Theories in the Finite Case	414
8.7	CONCLUDING REMARKS	417
8.7.1	Prescriptive Versus Descriptive Interpretations	417
8.7.2	Open Problems	420
	EXERCISES	420
 9. Measurement Inequalities		
□9.1	INTRODUCTION	423
9.2	FINITE LINEAR STRUCTURES	427
□9.2.1	Additivity	428
9.2.2	Probability	432
9.3	PROOF OF THEOREM 1	433
9.4	APPLICATIONS	434
9.4.1	Scaling Considerations	434
9.4.2	Empirical Examples	436
9.5	POLYNOMIAL STRUCTURES	447

9.6	PROOFS	450
9.6.1	Theorem 4	450
9.6.2	Theorem 5	451
9.6.3	Theorem 6	451
	EXERCISES	452
10.	Dimensional Analysis and Numerical Laws	
□10.1	INTRODUCTION	454
□10.2	THE ALGEBRA OF PHYSICAL QUANTITIES	459
10.2.1	The Axiom System	459
10.2.2	General Theorems	462
□10.3	THE PI THEOREM OF DIMENSIONAL ANALYSIS	464
10.3.1	Similarities	464
10.3.2	Dimensionally Invariant Functions	466
10.4	PROOFS	467
10.4.1	Preliminary Lemmas	467
10.4.2	Theorems 1 and 2	469
10.4.3	Theorem 3	470
10.4.4	Theorem 4	470
□10.5	EXAMPLES OF DIMENSIONAL ANALYSIS	471
10.5.1	The Simple Pendulum	472
10.5.2	Errors of Commission and Omission	474
10.5.3	Dimensional Analysis as an Aid in Obtaining Exact Solutions	479
10.5.4	Conclusion	480
□10.6	BINARY LAWS AND UNIVERSAL CONSTANTS	480
□10.7	TRINARY LAWS AND DERIVED MEASURES	483
10.7.1	Laws of Similitude	484
10.7.2	Laws of Exchange	488
10.7.3	Compatibility of the Trinary Laws	490
10.7.4	Some Relations Among Extensive, Difference, and Conjoint Structures	491
10.8	PROOFS	493
10.8.1	Preliminary Lemma	493
10.8.2	Theorem 5	494
10.8.3	Theorem 6	496
10.8.4	Theorem 7	498
□10.9	EMBEDDING PHYSICAL ATTRIBUTES IN A STRUCTURE OF PHYSICAL QUANTITIES	499
10.9.1	Assumptions About Physical Attributes	499
10.9.2	Fundamental, Derived, and Quasi-Derived Attributes	502

□10.10 WHY ARE NUMERICAL LAWS DIMENSIONALLY INVARIANT?	503
10.10.1 Three Points of View	503
10.10.2 Physically Similar Systems	506
10.10.3 Relations to Causey's Theory	512
10.11 PROOFS	513
10.11.1 Theorem 12	513
10.11.2 Theorem 13	515
10.12 INTERVAL SCALES IN DIMENSIONAL ANALYSIS	515
10.13 PROOFS	520
10.13.1 Preliminary Lemma	520
10.13.2 Theorem 14	521
10.13.3 Theorem 15	522
10.13.4 Theorem 16	523
10.14 PHYSICAL QUANTITIES IN MECHANICS AND GENERALIZATIONS OF DIMENSIONAL INVARIANCE	523
10.14.1 Generalized Galilean Invariance	526
10.14.2 Lorentz Invariance and Relativistic Mechanics	532
10.15 CONCLUDING REMARKS	535
EXERCISES	536
DIMENSIONS AND UNITS OF PHYSICAL QUANTITIES	539
Answers and Hints to Selected Exercises	545
References	551
<i>Author Index</i>	565
<i>Subject Index</i>	570

Table of Contents

<i>Preface</i>	xiii
<i>Acknowledgments</i>	xv

11. Overview

11.1	GEOMETRY UNIT	1
11.1.1	Geometrical Representations (Chapter 12)	3
11.1.2	Axiomatic Synthetic Geometry (Chapter 13)	4
11.1.3	Proximity Measurement (Chapter 14)	7
11.1.4	Color and Force Measurement (Chapter 15)	9
11.2	THRESHOLD AND ERROR UNIT	9
11.2.1	Representations with Thresholds (Chapter 16)	10
11.2.2	Representations of Choice Probabilities (Chapter 17)	11

12. Geometrical Representations

12.1	INTRODUCTION	13
12.2	VECTOR REPRESENTATIONS	14
12.2.1	Vector Spaces	16
12.2.2	Analytic Affine Geometry	21

12.2.3	Analytic Projective Geometry	24
12.2.4	Analytic Euclidean Geometry	31
12.2.5	Meaningfulness in Analytic Geometry	35
12.2.6	Minkowski Geometry	42
12.2.7	General Projective Metrics	46
12.3	METRIC REPRESENTATIONS	51
12.3.1	General Metrics with Geodesics	52
12.3.2	Elementary Spaces and the Helmholtz–Lie Problem	57
12.3.3	Riemannian Metrics	59
12.3.4	Other Metrics	71
	EXERCISES	77

13. Axiomatic Geometry and Applications

13.1	INTRODUCTION	80
13.2	ORDER ON THE LINE	83
13.2.1	Betweenness: Affine Order	83
13.2.2	Separation: Projective Order	85
13.3	PROOFS	89
13.4	PROJECTIVE PLANES	93
13.5	PROJECTIVE SPACES	102
13.6	AFFINE AND ABSOLUTE SPACES	104
13.6.1	Ordered Geometric Spaces	105
13.6.2	Affine Spaces	107
13.6.3	Absolute Spaces	109
13.6.4	Euclidean Spaces	110
13.6.5	Hyperbolic Spaces	111
13.7	ELLIPTIC SPACES	114
13.7.1	Double Elliptic Spaces	115
13.7.2	Single Elliptic Spaces	117
13.8	CLASSICAL SPACE-TIME	118
13.9	SPACE-TIME OF SPECIAL RELATIVITY	121
13.9.1	Other Axiomatic Approaches	127
13.10	PERCEPTUAL SPACES	131
13.10.1	Historical Survey through the Nineteenth Century	131
13.10.2	General Considerations Concerning Perceptual Spaces	134
13.10.3	Experimental Work before Luneburg's Theory	138
13.10.4	Luneburg Theory of Binocular Vision	139
13.10.5	Experiments Relevant to Luneburg's Theory	145
13.10.6	Other Studies	150
	EXERCISES	153

14. Proximity Measurement

14.1	INTRODUCTION	159
14.2	METRICS WITH ADDITIVE SEGMENTS	163
14.2.1	Collinearity	163
14.2.2	Constructive Methods	165
14.2.3	Representation and Uniqueness Theorems	168
14.3	PROOFS	169
14.3.1	Theorem 2	169
14.3.2	Reduction to Extensive Measurement	170
14.3.3	Theorem 3	171
14.3.4	Theorem 4	172
14.4	MULTIDIMENSIONAL REPRESENTATIONS	175
14.4.1	Decomposability	178
14.4.2	Intradimensional Subtractivity	179
14.4.3	Interdimensional Additivity	181
14.4.4	The Additive-Difference Model	184
14.4.5	Additive-Difference Metrics	185
14.5	PROOFS	188
14.5.1	Theorem 5	188
14.5.2	Theorem 6	189
14.5.3	Theorem 7	191
14.5.4	Theorem 9	192
14.5.5	Preliminary Lemma	194
14.5.6	Theorem 10	197
14.6	EXPERIMENTAL TESTS OF MULTIDIMENSIONAL REPRESENTATIONS	200
14.6.1	Relative Curvature	201
14.6.2	Translation Invariance	202
14.6.3	The Triangle Inequality	205
14.7	FEATURE REPRESENTATIONS	207
14.7.1	The Contrast Model	209
14.7.2	Empirical Applications	215
14.7.3	Comparing Alternative Representations	218
14.8	PROOFS	222
14.8.1	Theorem 11	222
	EXERCISES	224

15. Color and Force Measurement

15.1	INTRODUCTION	226
15.2	GRASSMANN STRUCTURES	229

15.2.1	Formulation of the Axioms	229
15.2.2	Representation and Uniqueness Theorems	234
15.2.3	Discussion of Proofs of Theorems 3 and 4	235
15.3	PROOFS	237
15.3.1	Theorem 3	237
15.3.2	Theorem 4	239
15.4	COLOR INTERPRETATIONS	240
15.4.1	Metameric Color Matching	240
15.4.2	Tristimulus Colorimetry	243
15.4.3	Four Ways to Misunderstand Color Measurement	250
15.4.4	Asymmetric Color Matching	252
□15.5	THE DIMENSIONAL STRUCTURE OF COLOR AND FORCE	255
15.5.1	Color Codes and Metamer Codes	256
15.5.2	Photopigments	260
15.5.3	Force Measurement and Dynamical Theory	263
15.5.4	Color Theory in a Measurement Framework	268
15.6	THE KÖNIG AND HURVICH–JAMESON COLOR THEORIES	271
15.6.1	Representations of 2-Chromatic Reduction Structures	271
15.6.2	The König Theory and Alternatives	277
15.6.3	Codes Based on Color Attributes	279
15.6.4	The Cancellation Procedure	280
15.6.5	Representation and Uniqueness Theorems	283
15.6.6	Tests and Extensions of Quantitative Opponent-Colors Theory	286
15.7	PROOFS	291
15.7.1	Theorem 6	291
15.7.2	Theorem 9	294
15.7.3	Theorem 10	295
	EXERCISES	296

16. Representations with Thresholds

16.1	INTRODUCTION	299
16.1.1	Three Approaches to Nontransitive Data	300
16.1.2	Idea of Thresholds	301
16.1.3	Overview	302
16.2	ORDINAL THEORY	303
16.2.1	Upper, Lower, and Two-Sided Thresholds	304
16.2.2	Induced Quasiorders: Interval Orders and Semiorders	306
16.2.3	Compatible Relations	311
16.2.4	Biorders: A Generalization of Interval Orders	313
16.2.5	Tight Representations	314
16.2.6	Constant-Threshold Representations	318
16.2.7	Interval and Indifference Graphs	322

16.3	PROOFS	326
16.3.1	Theorem 2	326
16.3.2	Lemma 1	326
16.3.3	Theorem 6	327
16.3.4	Theorem 9	327
16.3.5	Theorem 10	328
16.3.6	Theorem 11	329
16.3.7	Theorems 14 and 15	330
16.4	ORDINAL THEORY FOR FAMILIES OF ORDERS	331
16.4.1	Finite Families of Interval Orders and Semiorders	332
16.4.2	Order Relations Induced by Binary Probabilities	336
16.4.3	Dimension of Partial Orders	339
16.5	PROOFS	341
16.5.1	Theorem 16	341
16.5.2	Theorem 17	342
16.5.3	Theorem 18	343
16.5.4	Theorem 19	344
16.6	SEMIORDERED ADDITIVE STRUCTURES	344
16.6.1	Possible Approaches to Semiordered Probability Structures	345
16.6.2	Probability with Approximate Standard Families	349
16.6.3	Axiomatization of Semiordered Probability Structures	352
16.6.4	Weber's Law and Semiorders	353
16.7	PROOF OF THEOREM 24	354
16.8	RANDOM-VARIABLE REPRESENTATIONS	359
16.8.1	Weak Representations of Additive Conjoint and Extensive Structures	364
16.8.2	Variability as Measured by Moments	366
16.8.3	Qualitative Primitives for Moments	368
16.8.4	Axiom System for Qualitative Moments	371
16.8.5	Representation Theorem and Proof	374
	EXERCISES	379

17. Representation of Choice Probabilities

17.1	INTRODUCTION	383
17.1.1	Empirical Interpretations	384
17.1.2	Probabilistic Representations	385
17.2	ORDINAL REPRESENTATIONS FOR PAIR COMPARISONS	388
17.2.1	Stochastic Transitivity	388
17.2.2	Difference Structures	390
17.2.3	Local Difference Structures	391
17.2.4	Additive Difference Structures	393
17.2.5	Intransitive Preferences	397

17.3	PROOFS	401
17.3.1	Theorem 2	401
17.3.2	Theorem 3	405
17.3.3	Theorem 4	407
17.4	CONSTANT REPRESENTATIONS FOR MULTIPLE CHOICE	410
17.4.1	Simple Scalability	410
17.4.2	The Strict-Utility Model	413
17.5	PROOFS	419
17.5.1	Theorem 5	419
17.5.2	Theorem 7	420
17.6	RANDOM VARIABLE REPRESENTATIONS	421
17.6.1	The Random-Utility Model	421
17.6.2	The Independent Double-Exponential Model	424
17.6.3	Error Tradeoff	427
17.7	PROOFS	432
17.7.1	Theorem 9	432
17.7.2	Theorem 12	433
17.7.3	Theorem 13	435
17.8	MARKOVIAN ELIMINATION PROCESSES	436
17.8.1	The General Model	436
17.8.2	Elimination by Aspects	437
17.8.3	Preference Trees	444
17.9	PROOFS	450
17.9.1	Theorem 15	450
17.9.2	Theorem 16	452
17.9.3	Theorem 17	455
	EXERCISES	457
	References	459
	<i>Author Index</i>	481
	<i>Subject Index</i>	487

Table of Contents

<i>Preface</i>	xiii
<i>Acknowledgments</i>	xv
18. Overview	
18.1 NONADDITIVE REPRESENTATIONS (CHAPTER 19)	3
18.1.1 Examples	4
18.1.2 Representation and Uniqueness of Positive Operations	4
18.1.3 Intensive Structures	5
18.1.4 Conjoint Structures and Distributive Operations	6
18.2 SCALE TYPES (CHAPTER 20)	7
18.2.1 A Classification of Automorphism Groups	7
18.2.2 Unit Representations	9
18.2.3 Characterization of Homogeneous Concatenation and Conjoint Structures	10
18.2.4 Reprise	11
18.3 AXIOMATIZATION (CHAPTER 21)	11
18.3.1 Types of Axiom	11
18.3.2 Theorems on Axiomatizability	13
18.3.3 Testability of Axioms	14

18.4	INVARIANCE AND MEANINGFULNESS (CHAPTER 22)	14
18.4.1	Types of Invariance	15
18.4.2	Applications of Meaningfulness	16
19.	Nonadditive Representations	
19.1	INTRODUCTION	18
19.1.1	Inessential and Essential Nonadditives	18
19.1.2	General Binary Operations	22
19.1.3	Overview	23
19.2	TYPES OF CONCATENATION STRUCTURE	25
19.2.1	Concatenation Structures and Their Properties	25
19.2.2	Some Numerical Examples	27
19.2.3	Archimedean Properties	33
19.3	REPRESENTATIONS OF PCSs	37
19.3.1	General Definitions	37
19.3.2	Uniqueness and Construction of a Representation of a PCS	39
19.3.3	Existence of a Representation	40
19.3.4	Automorphism Groups of PCSs	44
19.3.5	Continuous PCSs	46
19.4	COMPLETIONS OF TOTAL ORDERS AND PCSs	48
19.4.1	Order Isomorphisms onto Real Intervals	49
19.4.2	Completions of Total Orders	50
19.4.3	Completions of Closed PCSs	53
19.5	PROOFS ABOUT CONCATENATION STRUCTURES	56
19.5.1	Theorem 1	56
19.5.2	Lemmas 1–6, Theorem 2	57
19.5.3	Theorem 2	61
19.5.4	Construction of PCS Homomorphisms	62
19.5.5	Theorem 3	64
19.5.6	Theorem 4	69
19.5.7	Theorem 5	70
19.5.8	Theorem 6	71
19.5.9	Corollary to Theorem 7	73
19.5.10	Theorem 9	73
19.6	CONNECTIONS BETWEEN CONJOINT AND CONCATENATION STRUCTURES	75
19.6.1	Conjoint Structures: Introduction and General Definitions	75
19.6.2	Total Concatenation Structures Induced by Conjoint Structures	77
19.6.3	Factorizable Automorphisms	79
19.6.4	Total Concatenation Structures Induced by Closed, Idempotent Concatenation Structures	81
19.6.5	Intensive Structures Related to PCSs by Doubling Functions	83
19.6.6	Operations That Distribute over Conjoint Structures	85

19.7	REPRESENTATIONS OF SOLVABLE CONJOINT AND CONCATENATION STRUCTURES	87
19.7.1	Conjoint Structures	87
19.7.2	Solvable, Closed, Archimedean Concatenation Structures	88
19.7.3	Intensive Concatenation Structures with Doubling Functions	89
19.8	PROOFS	89
19.8.1	Theorem 11	89
19.8.2	Theorem 12	92
19.8.3	Theorem 13	93
19.8.4	Theorem 14, Part (iii)	97
19.8.5	Theorem 15	97
19.8.6	Theorem 18	98
19.8.7	Theorem 21	100
19.9	BISYMMETRY AND RELATED PROPERTIES	101
19.9.1	General Definitions	101
19.9.2	Equivalences in Closed, Idempotent, Solvable, Dedekind Complete Structures	103
19.9.3	Bisymmetry in the 1-Point Unique Case	103
	EXERCISES	104

20. Scale Types

20.1	INTRODUCTION	108
20.1.1	Constructibility and Symmetry	108
20.1.2	Problem in Understanding Scale Types	111
20.2	HOMOGENEITY, UNIQUENESS, AND SCALE TYPE	112
20.2.1	Stevens' Classification	112
20.2.2	Decomposing the Classification	114
20.2.3	Formal Definitions	115
20.2.4	Relations among Structure, Homogeneity, and Uniqueness	117
20.2.5	Scale Types of Real Relational Structures	119
20.2.6	Structures with Homogeneous, Archimedean Ordered Translation Groups	122
20.2.7	Representations of Dedekind Complete Distributive Triples	125
20.3	PROOFS	126
20.3.1	Theorem 2	126
20.3.2	Theorem 3	127
20.3.3	Theorem 4	128
20.3.4	Theorem 5	128
20.3.5	Theorem 7	137
20.3.6	Theorem 8	141
20.4	HOMOGENEOUS CONCATENATION STRUCTURES	142
20.4.1	Nature of Homogeneous Concatenation Structures	142
20.4.2	Real Unit Concatenation Structures	143
20.4.3	Characterizations of Homogeneity: PCS	146

20.4.4	Characterizations of Homogeneity: Solvable, Idempotent Structures	148
20.4.5	Mixture Spaces of Gambles	150
20.4.6	The Dual Bilinear Utility Model	152
20.5	PROOFS	156
20.5.1	Theorem 9	156
20.5.2	Theorem 11	158
20.5.3	Theorem 24, Chapter 19	160
20.5.4	Theorem 14	163
20.5.5	Theorem 15	165
20.5.6	Theorem 16	166
20.5.7	Theorem 17	167
20.5.8	Theorem 18	171
20.5.9	Theorem 19	175
20.6	HOMOGENEOUS CONJOINT STRUCTURES	180
20.6.1	Component Homogeneity and Uniqueness	180
20.6.2	Singular Points in Conjoint Structures	181
20.6.3	Forcing the Thomsen Condition	183
20.7	PROOFS	184
20.7.1	Theorem 22	184
20.7.2	Theorem 23	185
20.7.3	Theorem 24	186
20.7.4	Theorem 25	190
	EXERCISES	192

21. Axiomatization

21.1	AXIOM SYSTEMS AND REPRESENTATIONS	196
21.1.1	Why Do Scientists and Mathematicians Axiomatize?	196
21.1.2	The Axiomatic-Representational Viewpoint in Measurement	201
21.1.3	Types of Representing Structures	202
21.2	ELEMENTARY FORMALIZATION OF THEORIES	204
21.2.1	Elementary Languages	204
21.2.2	Models of Elementary Languages	208
21.2.3	General Theorems about Elementary Logic	213
21.2.4	Elementary Theories	215
21.3	DEFINABILITY AND INTERPRETABILITY	218
21.3.1	Definability	218
21.3.2	Interpretability	224
21.4	SOME THEOREMS ON AXIOMATIZABILITY	225
21.5	PROOFS	229
21.5.1	Theorem 6	229
21.5.2	Theorem 7	230

TABLE OF CONTENTS

xi

21.5.3	Theorem 8	230
21.5.4	Theorem 9	231
21.6	FINITE AXIOMATIZABILITY	231
21.6.1	Axiomatizable by a Universal Sentence	234
21.6.2	Proof of Theorem 12	241
21.6.3	Finite Axiomatizability of Finitary Classes	242
21.7	THE ARCHIMEDEAN AXIOM	246
21.8	TESTABILITY OF AXIOMS	251
21.8.1	Finite Data Structures	254
21.8.2	Convergence of Finite to Infinite Data Structures	256
21.8.3	Testability and Constructibility	259
21.8.4	Diagnostic versus Global Tests	261
	EXERCISES	265

22. Invariance and Meaningfulness

22.1	INTRODUCTION	267
22.2	METHODS OF DEFINING MEANINGFUL RELATIONS	269
22.2.1	Definitions in First-Order Theories	271
22.2.2	Reference and Structure Invariance	273
22.2.3	An Example: Independence in Probability Theory	276
22.2.4	Definitions with Particular Representations	277
22.2.5	Parametrized Numerical Relations	278
22.2.6	An Example: Hooke's Law	280
22.2.7	A Necessary Condition for Meaningfulness	282
22.2.8	Irreducible Structures: Reference Invariance of Numerical Equality	284
22.3	CHARACTERIZATIONS OF REFERENCE INVARIANCE	285
22.3.1	Permissible Transformations	285
22.3.2	The Criterion of Invariance under Permissible Transformations	287
22.3.3	The Condition of Structure Invariance	287
22.4	PROOFS	290
22.4.1	Theorem 3	290
22.4.2	Theorem 4	290
22.4.3	Theorem 5	291
22.5	DEFINABILITY	292
22.6	MEANINGFULNESS AND STATISTICS	294
22.6.1	Examples	295
22.6.2	Meaningful Relations Involving Population Means	298
22.6.3	Inferences about Population Means	299
22.6.4	Parametric Models for Populations	299
22.6.5	Measurement Structures and Parametric Models for Populations	301
22.6.6	Meaningful Relations in Uniform Structures	305

22.7	DIMENSIONAL INVARIANCE	307
22.7.1	Structures of Physical Quantities	309
22.7.2	Triples of Scales	312
22.7.3	Representation and Uniqueness Theorem for Physical Attributes	315
22.7.4	Physically Similar Systems	318
22.7.5	Fundamental versus Index Measurement	323
22.8	PROOFS	326
22.8.1	Theorem 6	326
22.8.2	Theorem 7	327
22.9	REPRISE: UNIQUENESS, AUTOMORPHISMS, AND CONSTRUCTIBILITY	329
22.9.1	Alternative Representations	329
22.9.2	Nonuniqueness and Automorphisms	330
22.9.3	Invariance under Automorphisms	332
22.9.4	Constructibility of Representations	333
	EXERCISES	336
	References	338
	<i>Author Index</i>	347
	<i>Subject Index</i>	351